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Convergence Speed for Simple Symmetric Exclusion: An Explicit Calculation

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For the infinite-volume simple symmetric nearest-neighbor exclusion process in one dimension, we investigate the speed of convergence to equilibrium from a particular initial distribution. We use duality to reduce the analysis to that of the two-particle process, which we further reduce to a random walk reflecting rightward at zero, whose generator is self-adjoint on $l^2(Z)$. We obtain the spectral representation of the generator and use asymptotic analysis to show that convergence is slow.

KEY WORDS: Exclusion process; convergence speed; equilibrium; spectral representation.

1. INTRODUCTION

Within the field of interacting particle systems, there has long been interest in investigating speed of convergence to equilibrium. For the stochastic Ising model (see Holley,[9] Aizenman and Holley,[1] Martinelli, Olivieri, and Scoppola,[13] and Minlos and Trish[14]) and attractive reversible non-critical nearest particle systems (see Liggett[12] for the supercritical case and Mountford[15] for the subcritical), it is known that such convergence takes place exponentially fast.

In all the above processes, particles can be created and destroyed. In the exclusion process, though, particles are indestructible, their only interaction being to exclude one another from full sites. Due to the crowding among the particles, one might suspect that convergence for the exclusion process is much slower, at least if the jump range is not too large. In this paper, we prove this for one special case: the simple symmetric nearest-neighbor infinite-volume exclusion process in dimension $d = 1$. ("Simple"

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means that the maximum number of particles per site is 1, as opposed to more general N-exclusion.) Restriction to this case allows us to compute the spectral representation of a certain generator, to which we reduce the problem by using duality and the linear order of the integers. Slow convergence follows from the lack of a spectral gap. For other uses of this method, see Holley and Stroock[10] for certain one-dimensional stochastic Ising models and the aforementioned papers of Liggett[12] and Mountford.[15] (The method is mentioned by Deuschel and Stroock.[6] It also should be noted that Ferrari, Presutti, Scacciatelli, and Vares[8] have proven an upper bound on convergence speed using different methods.)

2. TWO-SITE CORRELATION AND DUALITY

In this paper, states of the system will be denoted by η and ζ, so that $\eta_t(x) \in \{0, 1\}$ is the number of particles at site x at time t. Ω denotes the infinitesimal generator, and $e^{t\Omega}$ is the Markov semigroup.

We start the process off with density 1/2 in the initial measure μ_0:

$$\mu_0\{\eta_{\text{even}}\} = \mu_0\{\eta_{\text{odd}}\} = 1/2 \tag{1}$$

where η_{even} is the state with a particle at every even site and no particles on the odd sites, and η_{odd} is the reverse. Thus the system initially has equal chances of being in either state.

This measure μ_0 will converge to some limiting measure μ_∞ (see Liggett,[11] Chapter 8). (In fact, Theorem 1.13 of that chapter may be used to show that the limiting measure is just the product measure with density 1/2, as might be expected. The conditions of the theorem follow from the main result of this paper, for which the nature of μ_∞ is irrelevant.)

In order to determine the speed of convergence, we investigate the probability that two arbitrary sites are both occupied. Let y and z be any two sites. Assume, without loss of generality, that $y < z$. We wish to study the correlation between $\eta_t(y)$ and $\eta_t(z)$, for which we have

$$\begin{aligned} P^{\mu_0}[\eta_t(y) = \eta_t(z) = 1] &= \mu_t\{\eta : \eta(y) = \eta(z) = 1\} \\ &\to \mu_\infty\{\eta : \eta(y) = \eta(z) = 1\} \end{aligned}$$

A similar correlation function was studied by Spohn[16] for the stochastic Ising model in one dimension with constant energy. We will see that for every value of t, $P^{\mu_0}[\eta_t(y) = \eta_t(z) = 1]$ is 1/4 plus an error term which goes to zero slowly.

We intend to use spectral theory to study the convergence rate. Even for the simple symmetric nearest-neighbor process, though, the generator is a complicated operator, and its spectral representation is difficult to find. Because of symmetry, however, we may use duality, which allows us to study the exclusion process with finitely many particles. Assuming symmetry and irreducibility of the jump probabilities $p(x, y)$, Theorem 1.1 of Liggett,[11] Chapter 8, states that for every η and every finite subset A of the site space S,

$$P^{\eta}[\eta_t = 1 \text{ on } A] = P^A[\eta = 1 \text{ on } A_t] \qquad \text{for all} \quad t \geqslant 0 \tag{2}$$

where A_t is the finite exclusion process with the same transition probabilities and $|A|$ particles. It may be helpful to think of running time backward in the dual process A_t: to see whether η_t "fills up" the set A, we start from particles only on A, run time backwards, and see whether A_t is completely contained in the occupied sites of the original η.

For our purposes, $A = \{y, z\}$, and we may use duality as follows:

$$\begin{aligned} P^{\mu_0}[\eta_t(y) = \eta_t(z) = 1] &= (1/2)(P^{\eta_{\text{even}}}[\eta_t = 1 \text{ on } A] + P^{\eta_{\text{odd}}}[\eta_t = 1 \text{ on } A]) \\ &= (1/2)(P^A[\eta_{\text{even}} = 1 \text{ on } A_t] + P^A[\eta_{\text{odd}} = 1 \text{ on } A_t]) \\ &= (1/2)(P^A[A_t \subset \{\text{evens}\}] + P^A[A_t \subset \{\text{odds}\}]) \\ &= (1/2)\, P^{\{y, z\}}[A_t \text{ has even separation}] \end{aligned}$$

3. THE SPECTRAL REPRESENTATION

We are now in a position to begin using spectral theory, since the generators we encounter are tractable enough. In the remainder of the paper, we take the Hilbert-space inner product $\langle f, g\rangle$ to be conjugate-linear in f rather than in g, according to the usual physicists' convention.

We formally define our two-particle process A_t as follows:

$$A_t = \{(X_1(t), X_2(t)) \colon \forall t\, X_1(t) < X_2(t)\};\ A_0 = \{y, z\}$$

Its generator Ω_A is bounded and so is defined on all of $l^2(X)$, where X is the state space. It is easy to show that Ω_A satisfies detailed balance with respect to counting measure, which implies that Ω_A is a symmetric (also called Hermitian) operator on the Hilbert space $l^2(X)$ (see Weidmann[18]). Since it is bounded as well, Ω_A is a self-adjoint operator on $l^2(X)$. Therefore the spectral theorem applies. To find a complete family of eigenfunctions of Ω_A, we can use the well-known solution of Bethe.[5] (A rigorous proof that Bethe's eigenfunctions form a complete family has been given by Babbitt and Thomas.[17, 2, 3, 4])

However, we may simplify the analysis by studying the difference between the positions of the two particles, reducing the problem to a reflecting random walk. If we study the process

$$Y_t = X_2(t) - X_1(t)$$

then Y_t is isomorphic to the one-particle process on $x \geq 1$, reflecting rightward at zero. Since

$$P^{z-y}[\,Y_t \text{ is even}\,] = P^{\{y,z\}}[\,A_t \text{ has even separation}\,]$$

we may investigate the rate of convergence of $P^{z-y}[\,Y_t \text{ is even}\,]$.

The transitions for the process Y_t have rate $\lambda/2$ to each side, except from $x = 1$, whence there is only a rate of $\lambda/2$ to the right. (Here $1/\lambda$ is the mean time a particle waits before it jumps in the two-particle process.) The process Y_t has generator Ω:

$$\Omega f(x) = \frac{\lambda}{2}\,[\,f(x+1) + f(x-1) - 2f(x)\,]$$

with boundary condition $f(0) = f(1)$. This generator is bounded and hence defined on all of $l^2(N)$, and it satisfies detailed balance with respect to counting measure. Again, this is enough to show that Ω is self-adjoint in $l^2(N)$. We will obtain its spectral representation to get that of the semigroup $e^{t\Omega}$.

The family $\Psi_k(x) = \sqrt{2/\pi}\cos(k(x-1/2))$ consists of eigenfunctions, as can easily be verified directly. (The factor of $\sqrt{2/\pi}$ normalizes them in $L^2[0, \pi]$.) These eigenfunctions are chosen so as to satisfy the boundary condition above; their eigenvalues are $\omega(k) = \lambda(\cos(k) - 1)$.

For $f \in l^1(N)$, define the operator U as follows:

$$(Uf)(k) = \sum_{x=1}^{\infty} \Psi_k(x)\, f(x) \tag{3}$$

Then U is clearly bounded and therefore has a unique bounded extension to all of $l^2(N)$. Define the operator V on $L^2[0, \pi]$ thus:

$$(Vh)(x) = \langle \Psi_k(x), h \rangle = \int_0^{\pi} \Psi_k(x)\, h(k)\, dk \tag{4}$$

By Bessel's inequality, V is bounded, and it can be shown that U is a Hilbert space isomorphism with $V = U^{-1}$.

Writing $\hat{f}$ for Uf, we have that for every $f \in l^2(N)$,

$$f(x) = \int_0^\pi \Psi_k(x)\, \hat{f}(x)\, dk \tag{5}$$

This means that for every such f,

$$(\Omega f)(x) = \int_0^\pi \Psi_k(x)\, \omega(k)\, \hat{f}(k)\, dk \tag{6}$$

By self-adjointness, the generator $e^{t\Omega}$ has the same spectral representation, with $\omega(k)$ replaced by $e^{t\omega(k)}$. That is, for every $f \in l^2(N)$,

$$(e^{t\Omega} f)(x) = \int_0^\pi \Psi_k(x)\, e^{t\omega(k)} \hat{f}(k)\, dk \tag{7}$$

4. THE CONVERGENCE STUDY

We now wish to investigate the rate of convergence of $P^{z-y}[\,Y_t$ is even$]$. To do so, let $f(x)$ be the indicator function of the even integers 2, 4, 6,..., and for $r \in (0, 1)$, let $f_r(x) = r^x f(x) \in l^1(N)$. Then we have

$$\begin{aligned} P^{z-y}[\,Y_t \text{ is even}\,] &= E^{z-y}[\,f(Y_t)\,] && (8)\\ &= \lim_{r \nearrow 1} E^{z-y}[\,f_r(Y_t)\,] && (9)\\ &= \lim_{r \nearrow 1} (e^{t\Omega} f_r)(z-y) && (10)\\ &= \lim_{r \nearrow 1} \int_0^\pi \Psi_k(z-y)\, e^{t\omega(k)} \hat{f}_r(k)\, dk && (11) \end{aligned}$$

where (9) follows from the Dominated Convergence Theorem. To compute the transform $\hat{f}_r(k)$, it is convenient to use the identity

$$2\cos(k(x-1/2)) = e^{ikx} e^{-ik/2} + e^{-ikx} e^{ik/2}$$

This allows one to use the very useful device of computing the sum of a geometric series.

By a routine calculation, we obtain

$$\begin{aligned} \sqrt{\pi/2}\, \hat{f}_r(k) &= (1/2) \left[\frac{e^{-ik/2} r^2 e^{2ik}}{1 - r^2 e^{2ik}} + \frac{e^{ik/2} r^2 e^{-2ik}}{1 - r^2 e^{-2ik}} \right] \\ &= (1/2) \left[\frac{(r^2 e^{ik} - r^4 e^{-ik})\, e^{ik/2}}{1 - 2r^2 \cos(2k) + r^4} + \frac{(r^2 e^{-ik} - r^4 e^{ik})\, e^{-ik/2}}{1 - 2r^2 \cos(2k) + r^4} \right] \end{aligned}$$

We can rewrite this as

$$\sqrt{\pi/2}\,\hat{f}_r(k)=\frac{r^2(1-r^2)\cos(k)\cos(k/2)}{1-2r^2\cos(2k)+r^4}-\frac{r^2(1+r^2)\sin(k)\sin(k/2)}{1-2r^2\cos(2k)+r^4} \tag{12}$$

From (11), using arbitrary x in place of $z-y$, we now have

$$\begin{aligned}
&\int_0^\pi \Psi_k(x)\, e^{t\omega(k)}\hat{f}_r(k)\,dk\\
&\quad=\int_0^\pi \sqrt{2/\pi}\cos(k(x-1/2))\,e^{\lambda t(\cos k-1)}\hat{f}_r(k)\,dk\\
&\quad=\int_0^\pi \cos(k(x-1/2))\,e^{\lambda t(\cos k-1)}\cos(k)\cos(k/2)\,P_r(k)\frac{2}{\pi}\,dk \qquad (13)\\
&\qquad-\int_0^\pi e^{\lambda t(\cos k-1)}g_r(k)\frac{2}{\pi}\,dk \qquad (14)
\end{aligned}$$

where

$$P_r(k)=\frac{r^2(1-r^2)}{1-2r^2\cos(2k)+r^4} \tag{15}$$

$$g_r(k)=\frac{r^2(1+r^2)\cos(k(x-1/2))\sin(k)\sin(k/2)}{1-2r^2\cos(2k)+r^4} \tag{16}$$

We now investigate the limits of these integrals as $r\to 1$. The first gives the limiting correlation value of 1/4; the second gives an error term through which we will see the convergence speed.

4.1. The Limiting Value

For (13), $P_r(k)$ will turn out to give the sum of two delta functions in the limit, as we might expect since it is similar to the Poisson kernel. To see this, we need three propositions about $P_r(k)$; the proofs are left to the reader. The arguments parallel the standard derivation of the Poisson kernel.

Proposition 1. For every $r\in(0,1)$, $P_r(k)$ is nonnegative for $k\in[0,\pi]$.

Proposition 2. On any k-interval $[\varepsilon,\pi-\varepsilon]$, $P_r(k)\to 0$ uniformly in k.

Proposition 3.

$$\int_0^{\pi/2} P_r(k)\,dk = \int_{\pi/2}^{\pi} P_r(k)\,dk = \frac{\pi r^2}{2(1+r^2)} \tag{17}$$

Lemma 1. $\forall f \in C(R)$,

$$\lim_{r \nearrow 1} \int_0^{\pi/2} P_r(k)\, f(k)\,dk = \frac{\pi}{4} f(0)$$

Proof. We have

$$\frac{\pi}{4} f(0) = \frac{\pi}{4}\frac{2(1+r^2)}{\pi r^2} \int_0^{\pi/2} P_r(k)\, f(0)\,dk$$

Thus,

$$\left|\int_0^{\pi/2} P_r(k)\, f(k)\,dk - \frac{\pi}{4} f(0)\right| \tag{18}$$

$$= \left|\int_0^{\pi/2} P_r(k)\, f(k)\,dk - \frac{(1+r^2)}{2r^2}\int_0^{\pi/2} P_r(k)\, f(0)\,dk\right| \tag{19}$$

$$\leqslant \int_0^{\pi/2} P_r(k) \left| f(k) - \frac{(1+r^2)}{2r^2} f(0)\right| dk \tag{20}$$

$$\approx \int_0^{\pi/2} P_r(k)\, |f(k) - f(0)|\,dk \tag{21}$$

when r is near 1. (This can easily be made rigorous since $P_r(k)$ has a finite integral on $[0, \pi/2]$.) Since f is continuous, we can split $[0, \pi/2]$ into $[0, \delta]$ and $[\delta, \pi/2]$ such that $|f(k) - f(0)|$ is small on the first interval and $|P_r(k)|$ is uniformly small on the second. Hence we can make our expression as small as desired by taking r close enough to 1, which proves the lemma. ∎

Lemma 2. $\forall f \in C(R)$,

$$\lim_{r \nearrow 1} \int_{\pi/2}^{\pi} P_r(k)\, f(k)\,dk = \frac{\pi}{4} f(\pi)$$

Proof. The above proof goes straight through, *mutatis mutandis.* ∎

These two lemmas show that in the limit, $P_r(k)$ gives the distribution

$$(\pi/4)\,\delta(k) + (\pi/4)\,\delta(k-\pi)$$

We can now find the limit of (13). Let

$$\tilde{f}_r(k) = \cos(k(x-1/2))\, e^{\lambda t(\cos k - 1)} \cos(k) \cos(k/2)\, \frac{2}{\pi}$$

Then $\tilde{f}_r(k) \in C(R)$, and therefore

$$\begin{aligned}
&\lim_{r \nearrow 1} \int_0^{\pi} \tilde{f}_r(k)\, P_r(k)\, dk \\
&\quad = \lim_{r \nearrow 1} \left[\int_0^{\pi/2} \tilde{f}_r(k)\, P_r(k)\, dk + \int_{\pi/2}^{\pi} \tilde{f}_r(k)\, P_r(k)\, dk \right] && (22) \\
&\quad = \lim_{r \nearrow 1} \int_0^{\pi/2} \tilde{f}_r(k)\, P_r(k)\, dk + \lim_{r \nearrow 1} \int_{\pi/2}^{\pi} \tilde{f}_r(k)\, P_r(k)\, dk && (23) \\
&\quad = (\pi/4)\, \tilde{f}_r(0) + (\pi/4)\, \tilde{f}_r(\pi) = 1/2 && (24)
\end{aligned}$$

4.2. The Error Term

For the second integral, (14), we seek to find

$$\lim_{r \nearrow 1} \int_0^{\pi} e^{\lambda t(\cos k - 1)} g_r(k)\, \frac{2}{\pi}\, dk \tag{25}$$

where, again,

$$g_r(k) = \frac{r^2(1+r^2)\cos(k(x-1/2))\sin(k)\sin(k/2)}{1 - 2r^2\cos(2k) + r^4}$$

For $r \in (0, 1)$, $g_r(k) \in C(R)$, and for all $k \notin \{0, \pi\}$,

$$\lim_{r \nearrow 1} g_r(k) = \frac{\cos(k(x-1/2))\sin(k)\sin(k/2)}{1-\cos(2k)} \equiv g(k)$$

This limiting function $g(k)$ has two singularities, at 0 and π, but it is easy to check that they are removable, and so we may extend $g(k)$ to be continuous on $[0, \pi]$. (In fact, $g(0) = 1/4$ independently of x.) Then $g_r \to g$ a.e. on $[0, \pi]$. It is not difficult to show that $g_r(k)$ has a uniform bound, independent of both k and r. We may therefore use the Dominated Convergence Theorem on the above integral to obtain

$$\lim_{r \nearrow 1} \int_0^\pi e^{\lambda t(\cos k - 1)} g_r(k) \frac{2}{\pi}\, dk = \int_0^\pi e^{\lambda t(\cos k - 1)} g(k) \frac{2}{\pi}\, dk \tag{26}$$

4.3. The Rate of Convergence

Putting (24) and (26) together allows us to express the probability under discussion as follows:

$$\begin{aligned}
&P^{\mu_0}[\eta_t(y) = \eta_t(z) = 1]\\
&\quad = (1/2)\, P^{z-y}[\,Y_t \text{ is even}]\\
&\quad = (1/2) \lim_{r \nearrow 1} (e^{t\Omega} f_r)(z-y)\\
&\quad = (1/2) \lim_{r \nearrow 1} \left[\int_0^\pi \cos(k(z-y-1/2))\, e^{\lambda t(\cos k - 1)} \right.\\
&\qquad \times \cos(k) \cos(k/2)\, P_r(k) \frac{2}{\pi}\, dk\\
&\qquad \left. - \int_0^\pi e^{\lambda t(\cos k - 1)} g_r(k) \frac{2}{\pi}\, dk \right]\\
&\quad = 1/4 - (1/2) \int_0^\pi e^{\lambda t(\cos k - 1)} g(k) \frac{2}{\pi}\, dk
\end{aligned}$$

We see that the first part of the spectral expansion remains at 1/4 for all t, and the second part gives an error term depending on t. To see how this error behaves as $t \to \infty$, we use Laplace's method (see, for instance, Erdélyi,[7] pp. 36–37). Let

$$\tilde{g}(k) = (2/\pi)\, g(k)$$

and

$$h(k) = \lambda(\cos k - 1)$$

Our integral becomes

$$\int_0^\pi \tilde{g}(k)\, e^{th(k)}\, dk$$

Then $\tilde{g}$ is continuous, h is infinitely differentiable, and we at once have that $h'(0)=0$, $h''(0)=-\lambda<0$, and on $[0,\pi]$, h attains its maximum at $k=0$ and nowhere else. Hence we have Laplace's result:

$$\int_0^\pi \tilde{g}(k)\, e^{th(k)}\, dk \sim \tilde{g}(0)\, e^{th(0)} \left[\frac{-\pi}{2th''(0)}\right]^{1/2} \tag{27}$$

$$=(2/\pi)(1/4)\left[\frac{\pi}{2t\lambda}\right]^{1/2} \tag{28}$$

$$=\frac{1}{2\sqrt{2\pi\lambda t}} \qquad \text{as} \quad t\to\infty \tag{29}$$

where $g(0)=1/4$ follows from L'Hôpital's Rule.

Therefore, we have shown that as $t\to\infty$,

$$P^{\mu_0}[\eta_t(y)=\eta_t(z)=1]\sim\frac{1}{4}-\frac{1}{4\sqrt{2\pi\lambda t}} \tag{30}$$

Thus, as suspected, convergence to equilibrium occurs quite slowly in this case, and the asymptotic rate of convergence does not depend on the separation of the arbitrary sites y and z.

(It is interesting to note that for the simple random walk on Z, without the reflecting barrier at 0, the probability of being on an even integer goes to 1/2 exponentially fast. This makes the above result for the reflected walk somewhat surprising. The barrier not only makes the rate polynomial, but a rather slow polynomial as well. What this might mean in terms of the exclusion process is left to the reader to judge.)

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REFERENCES

1. M. Aizenman and R. Holley, Rapid convergence to equilibrium of stochastic Ising models in the Dobrushin-Schlosman regime, in *Percolation Theory and Ergodic Theory of Infinite Particle Systems*, H. Kesten, ed., (Springer, New York, 1989, pp. 1-11).

2. D. Babbitt and L. E. Thomas, Ground state representation of the infinite one-dimensional Heisenberg ferromagnet, II. An explicit Plancherel formula, *Commun. Math Phys.* **54**:255–278 (1977).
3. D. Babbitt and L. E. Thomas, Ground state representation of the infinite one-dimensional Heisenberg ferromagnet, III. Scattering theory, *J. Math. Phys.* **19**:1699–1704 (1978).
4. D. Babbitt and L. E. Thomas, Ground state representation of the infinite one-dimensional Heisenberg ferromagnet, IV. A completely integrable quantum system. *J. Math. Analysis Appl.* **72**:305–328 (1979).
5. H. M. Bethe, Zur Theorie der Metalle. I. Eigenwerte und Eigenfunktionen der linearen Atomkette, *Z. Phys.* **71**:205–226 (1931).
6. J. Deuschel and D. W. Stroock, Hypercontractivity and spectral gap of symmetric diffusions with applications to the stochastic Ising models, *J. Functional Analysis* **92**:30–48 (1990).
7. A. Erdélyi, *Asymptotic Expansions* (Dover Publications, Inc., New York, 1956).
8. P. A. Ferrari, E. Presutti, E. Scacciatelli, and M. E. Vares, The symmetric simple exclusion process, I: probability estimates, *Stoch. Proc. Appl.* **39**:89–105 (1991).
9. R. Holley, Rapid convergence to equilibrium in one-dimensional stochastic Ising models, *Ann. Prob.* **13**:72–89 (1985).
10. R. Holley and D. Stroock, Uniform and L^2 convergence in one dimensional stochastic Ising models, *Comm. Math. Phys.* **123**:85–93 (1989).
11. T. M. Liggett, *Interacting Particle Systems* (Springer-Verlag, New York, 1985).
12. T. M. Liggett, Exponential L^2 convergence of attractive reversible nearest particle systems, *Ann. Prob.* **17**:403–432 (1989).
13. F. Martinelli, E. Olivieri, and E. Scoppola, Metastability and exponential approach to equilibrium for low-temperature stochastic Ising models, *J. Stat. Phys.* **61**:1105–1119 (1990).
14. R. A. Minlos and A. G. Trish, Polnoye spektralnoe razlozhenye generatora Glauberovoy dinamiki dlya odnomernoy modeli Isinga, *Uspyechi Matematicheskych Nauk* (Russian Academy of Sciences, Moscow, 1994).
15. T. S. Mountford, Exponential convergence for attractive reversible subcritical nearest particle systems, *Stoch. Proc. Appl.* **59**:235–249 (1995).
16. H. Spohn, Stretched exponential decay in a kinetic Ising model with dynamical constraint, *Comm. Math. Phys.* **125**:3–12 (1989).
17. L. E. Thomas, Ground state representation of the infinite one-dimensional Heisenberg ferromagnet, I, *J. Math. Analysis Appl.* **59**:392–414 (1977).
18. J. Weidmann, *Linear Operators in Hilbert Spaces* (Springer-Verlag, New York, 1980).